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Padé summability of the cubic oscillator

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Abstract
We prove the Padé (Stieltjes) summability of the perturbation series of any energy level $E_{n,1}(\beta)$, $n \in \mathbb{N}$, of the cubic anharmonic oscillator, $H_1(\beta) = p^2 + x^2 + \mathrm{i}\sqrt{\beta}x^3$, as suggested by the numerical studies of Bender and Weniger. At the same time, we give a simple proof of the positivity of every level of the $\mathcal{PT}$-symmetric Hamiltonian $H_1(\beta)$ for positive β (Bessis–Zinn Justin conjecture). The n zeros, of a state $\psi_{n,1}(\beta)$, stable at $\beta = 0$, are confined for β on the cut complex plane, and are related to the level $E_{n,1}(\beta)$ by the Bohr–Sommerfeld quantization rule (semiclassical phase-integral condition). We also prove the absence of non-perturbative eigenvalues and the simplicity of the spectrum of our Hamiltonians.

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1. Introduction

The anharmonic oscillators, and in particular the cubic one, are non-solvable quantum models, interesting due to their simplicity. The Hamiltonians considered have compact resolvents and their spectrum is discrete. New interest comes from the theory of the $\mathcal{PT}$-symmetric Hamiltonians. In particular, the interest is addressed to the summability of the perturbation series, also in connection to similar problems in quantum field theory.

Many years ago [1], Padé summability (PS) of the perturbation series of the energy levels of the quartic anharmonic oscillator with Hamiltonian $K_{4,1}(\beta) = p^2 + x^2 + \beta x^4$ was proved.

Some years later [2], the Borel summability of the perturbation series of each eigenvalue $E_{n,\alpha}(\beta)$, $n \in \mathbb{N}$, of the cubic anharmonic oscillator,

$$H_\alpha(\beta) = p^2 + \alpha x^2 + \mathrm{i}\sqrt{\beta}x^3, \qquad (1)$$

for a fixed $\alpha > 0$, was proved. This result was later extended [3], giving the distributional Borel summability [4] of the perturbation series, in the case of negative β.

The conjecture of Bessis–Zinn Justin (BZJ) was proved by Dorey *et al* [12] at $\alpha = 0$. Shin [5] extended the proof to $\alpha \in \mathbb{R}$, and proved the positivity of the eigenvalues $\{E_{n,\alpha}(\beta)\}_n$ for $\alpha \geqslant 0$, $\beta > 0$. Strangely enough, Bessis did not suggest, as far as we know, that the reality of the eigenvalues was a consequence of his loved PS. Some years later, Bender and Weniger gave numerical evidence of PS [9].

The BZJ conjecture was later extended by Bender and Boettcher (BB) [7], to the family of $\mathcal{PT}$-symmetric (PTS) Hamiltonians,

$$H_{N,\alpha}(1) = p^2 + \alpha x^2 - (\mathrm{i}x)^N,$$

$\alpha \geqslant 0$, $N \geqslant 2$, with analytic eigenfunctions $\phi(z)$, where $z = x + \mathrm{i}y$, vanishing at infinity on the two Stokes angular sectors of the complex plane,

$$S^N_{\pm 1} = \left\{ \left| \arg(\mathrm{i}z) \pm \frac{2\pi}{(N+2)} \right| < \frac{\pi}{(N+2)} \right\}. \tag{2}$$

The last conjecture was proved, as part of a more general result, by Vladimir Bouslaev and one of us [10] (see also [11]), in the relevant case of $N = 4$.

Shin has proved the BB conjecture, in the general case, for $\alpha \leqslant 0$ [5].

From now on, we restrict the discussion to the cubic oscillator. The family of operators $H_1(\beta)$ is an analytic family of type A on the cut plane $C_c = \{\beta \in \mathbb{C}; \beta \neq 0, |\arg(\beta) = \theta| < \pi\}$, and we have the spectral equivalence [13],

$$H_1(\beta) \sim \alpha^{-1/2} H_\alpha(1), \tag{3}$$

where $\alpha = \beta^{-2/5}$. For β at the hedges of the cut, for instance, $\beta = b\exp(-\mathrm{i}\pi) = -b - \mathrm{i}0^+$, $b > 0$, the mechanical problem defined by the formal Hamiltonian $H_1(\beta)$ is incomplete in both classical and quantum cases and can be defined by the physical hypothesis of the disappearance of the particle when it reaches infinity. In the quantum case, this means defining the Hamiltonian by the Gamow condition at $-\infty$ [2]. This definition satisfies the condition of subdominant behavior on the sectors $S_{\pm 1}(-\pi)$ (6), and the continuity in $\beta \in \bar{C}_c = \{\beta \in \mathbb{C}; \beta \neq 0, |\arg(\beta) = \theta| \leqslant \pi\}$, extending the unitary equivalence (3). The eigenvalues have the meaning of resonances and the eigenfunctions have the meaning of metastable states for the dynamical problem. Thus, we expect, and we prove, a negative imaginary part of the eigenvalues, related, in the usual way, to the lifetime of the metastable states.

We consider the eigenfunction $\psi_{n,\alpha,\beta}(z)$, for a fixed $\alpha > 0$, where $n \in \mathbb{N}$ is the number of its nodes, and β is on the complex cut plane C_c. The n nodes, numerically studied in [8] for positive β, are stable at $\beta = 0$ and are the only zeros on the lower half complex plane $\mathbb{C}_- = \{z \in \mathbb{C}; \Im(z) = y < 0\}$. On the other side [6], there are no zeros on the strip $0 \leqslant \Im(z) \leqslant y_+ = 2\alpha\Re\sqrt{\beta}/3b$.

We use the Loeffel–Martin method and the complex semiclassical Sibuya picture [6, 14], to prove the confinement of the nodes. The eigenvalues are bounded, in the suitable scaling, because of the Bohr–Sommerfeld quantization rule (24), (25) and the invariance of the number of nodes. This fact forbids both the disappearance at infinity of the perturbative eigenvalues and the appearance of non-perturbative eigenvalues at a non-zero parameter β. The total exclusion of non-perturbative eigenvalues comes from the existence of only one point $z^d \neq 0$, where the potential is stationary $V'(z^d) = 0$. We prove that the top resonances localized near this point are not the eigenvalues of our Hamiltonian.

The crossings of eigenvalues and the branch-point singularities are forbidden by the unique characterization of the eigenfunctions by the number of their nodes, and the spectrum is simple in both the geometric and algebraic sense (see [16] vol. IV).

This means that the generalized eigenvectors are ordinary eigenvectors.

Let us remember that we have the extended $\mathcal{PT}$ symmetry (see [2]) of the complex Hamiltonians,

$$H_1(\beta) = \mathcal{P}H_1^*(\bar{\beta})\mathcal{P}, \qquad \text{where} \quad \mathcal{P}\psi(x) = \psi(-x).$$

The isolation and analyticity of each eigenvalue on the cut plane $\mathbb{C}_c$ and the unique sum of the perturbation series imply the extended $\mathcal{PT}$ symmetry of the eigenfunctions, $\psi_{n,1,\beta}(x) = \bar{\psi}_{n,1,\bar{\beta}}(-x)$, and eigenvalues $E_{n,1}(\beta) = \bar{E}_{n,1}(\bar{\beta})$. The identity, obtained by complex scaling for $\beta \neq 0$, $|\arg(\beta)| < \pi$,

$$\{E_{n,1}(\beta) = \alpha^{-1/2}E_{n,\alpha}(1)\}_{n\in\mathbb{N}}, \tag{4}$$

where $\alpha = \beta^{-2/5}$ allows the global analytic continuation on the Riemann surface of $\beta^{1/5}$, of the set of the eigenvalues. In particular, we prove the power law behavior of the eigenvalues at $\beta = \infty$ by the scaling law (4) and the analyticity of $\{E_{n,\alpha}(1)\}_n$ at $\alpha = 0$.

We prove the PS of the perturbation series of each eigenvalue to the eigenvalue itself. In order to be more precise, let us fix $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, and set the simplified notations for the once subtracted eigenvalue, $f(\beta) = (E_{n,1}(\beta) - E_{n,1}(0))/\beta$, for any β on the cut complex plane $\mathbb{C}_c$. Thus, for $\beta \in \mathbb{C}_c$, we have the Stieltjes representation for $f(\beta)$, and the asymptotics for small $b = |\beta|$ given by the formal perturbation series [2]

$$f(\beta) = \int_0^\infty \frac{1}{(1+\beta\lambda)}\rho(\lambda)\,\mathrm{d}\lambda \sim \Sigma(\beta) = \sum_{k=0}^\infty c_{k+1}\beta^k,$$

where $\rho(\lambda)$ is non-negative, and $\{c_j\}_{j\in\mathbb{N}}$ are the perturbation coefficients of $E_{n,1}(\beta)$.

Thus, we prove, in a new way, the positivity of the eigenvalues, for positive β,

$$E_{n,1}(\beta) = E_{n,1}(0) + \beta f(\beta) = E_{n,1}(0) + \beta\int_0^\infty \frac{1}{(1+\beta\lambda)}\rho(\lambda)\,\mathrm{d}\lambda \geqslant E_{n,1}(0) > 0.$$

The PS of the perturbation series to the eigenvalue is defined by the limit

$$f(\beta) = \lim_{k\to\infty} R_k^k(\beta),$$

where $R_k^k(\beta) = P_k(\beta)/Q_k(\beta)$ are the diagonal Padé approximants, $P_k(\beta)$, $Q_k(\beta)$ are the polynomials of order k, with $Q_k(0) = 1$, completely defined by the asymptotics for $|\beta|$ small

$$\left|R_k^k(\beta) - \Sigma^{2k+1}(\beta)\right| = O(|\beta|^{2k+1}),$$

where $\Sigma^k(\beta) = \sum_{j=0}^{k-1} c_{j+1}\beta^j$.

The semiclassical behavior, for large positive λ, of the discontinuity

$$\ln(\rho(\lambda)) = -C^{-1}\lambda(1 + O(\ln(\lambda)/\lambda)),$$

where $C = 15/8$ agrees with the asymptotics of the perturbation coefficients for large j, as computed in [9], for $n = 0$:

$$c_j = (-1)^{j+1}4\sqrt{15}C^j(2\pi)^{-3/2}\Gamma(j+1/2)(1 + O(1/j)).$$

For numerical aspects, as the interesting similarity of this perturbation series with one of the quartic anharmonic oscillator, see [9].

In section 2 we consider the stability, analyticity and asymptotics of the eigenvalues and the nodes of the eigenfunctions for small $|\beta|$. In section 3 we confine the nodes on the lower complex half plane and we extend the results of section 2. In section 4 we prove the stability of the nodes for a small parameter. In section 5 we prove the stability of the nodes for a large parameter. In section 6 we prove the limitation of an eigenvalue for bounded parameters. In section 7 we prove the absence of non-perturbative eigenvalues. In section 8 we prove the power law behavior at infinity in the parameter. In section 9 we prove the Padé summability of the perturbation series.

2. Stability of the nodes at $\beta = 0$

From now on, we call $b = |\beta|$, and, for $b > 0$, $\theta = \arg(\beta)$, so that

$$\beta = b\exp(\mathrm{i}\theta).$$

For symmetry reasons we can restrict ourself to negative θ, $-\pi \leqslant \theta \leqslant 0$, (the case $0 \leqslant \theta \leqslant \pi$ is equivalent). Let us consider the analytic family of type A of compact resolvent operators,

$$H_\alpha(\beta), \tag{5}$$

on the domain $D = D(p^2) \cap D(x^3)$ for fixed $\alpha \in \mathbb{C}$, β on the cut plane

$$\mathbb{C}_c = \{\beta \in \mathbb{C}; b > 0, |\theta| < \pi, \}$$

([2], theorem 2.9).

We fix, for example, $\alpha = 1$. The eigenvalue $E_{n,1}(\beta)$, for a fixed $n = 0, 1, \ldots$ of $H_1(\beta)$, is also an eigenvalue of the operator $\alpha^{1/2}H_\alpha(b)$, $E_{n,1}(\beta) = \alpha^{1/2}E_{n,\alpha}(b)$ (the index n is the number of stable zeros (nodes) of the eigenfunction) where

$$\alpha = (b/\beta)^{2/5} = \exp(-2\mathrm{i}\theta/5).$$

In particular $E_{n,1}(b\exp(\pm\pi)) = \sqrt{\alpha}E_{n,\alpha}(b)$, where $\alpha = \exp(\mp 2\mathrm{i}\pi/5)$.

Let us recall [14] the five Stokes angular sectors of the complex z plane, for $\beta \neq 0$,

$$S_k = S_k(\theta) = \left\{z \in \mathbb{C};\left|\arg(\mathrm{i}z) + \frac{\theta}{10} - \frac{2k\pi}{5}\right| < \frac{\pi}{5}\right\}, \tag{6}$$

$-2 \leqslant k \leqslant 2$. In the case of $\theta = -\pi$,

$$S_{-1}(\theta) = \left\{z \in \mathbb{C}; -\pi < \arg(z) < -\frac{3\pi}{5}\right\},$$

where $\arg(z) = -\pi$, that is $x < 0$ is on the upper border of the sector, where the subdominant solution on the sector satisfies the Gamow condition.

For β on the closed cut plane, $\overline{\mathbb{C}}_c = \{\beta \in \mathbb{C}; b > 0, |\theta| \leqslant \pi\}$, we have the spectral equivalence for scaling:

$$H_\alpha(b) \sim (\alpha)^{-1/2}H_1(\beta), \tag{7}$$

where $\alpha = \exp(-2\mathrm{i}\theta/5)$ if $H_1(\beta)$, at $\theta = -\pi$, is defined by the Gamow condition at $-\infty$.

In place of the limit of $H_1(\beta)$, as $\beta \to 0$, we consider the norm resolvent limit $H_\alpha(b) \to H_\alpha(0)$, for $\alpha = \exp(-2\mathrm{i}\theta/5)$ fixed, as $b \to 0$. Let us note that $H_\alpha(0)$, for $\alpha \neq 0$, is defined on the domain $D = D(p^2) \cap D(x^2)$ (see theorem 2.13 on [2], and its extension on [3]).

We have the result of stability, simplicity and strong asymptotics of the eigenvalues:

Theorem 1. *For $n = 0, 1, \ldots$ let $E_{n,1}(\beta)$ be an eigenvalue, and let $\{c_k\}_{k\in\mathbb{N}}$ be its perturbation coefficients,*

$$f(\beta) = \frac{(E_{n,1}(\beta) - E_{n,1}(0))}{\beta}.$$

Then, there exists $b_n > 0$ such that $f(\beta)$ is analytic on the bounded sector,

$$\Omega_n = \{\beta \in \mathbb{C}; 0 < |\beta| < b_n, |\arg(\beta)| \leqslant \pi\},$$

and there exist numbers $A, C > 0$, such that

$$|f(\beta) - \Sigma^N(\beta)| < (AC^N N!|\beta|^N),$$

where $\Sigma^N(\beta) = \sum_{k=0}^{N-1} c_k(\beta)^k$, uniformly for $N - 1 \in \mathbb{N}$ and $\beta \in \Omega_n$.

Proof. See [2], theorem 3.2 (where β is our $\mathrm{i}\sqrt{\beta}$) extended in [3] (where our case is at $k = 1$). □

Lemma 1: The stability of the nodes. *Together with the stability of the eigenvalues, we have the stability of the eigenfunctions. In particular, we are interested in the stability of their zeros (nodes) at* $\beta = 0$.

We have the limit of the eigenvalue $E_{n,1}(\beta) \to E_{n,1}(0)$ *and the strong limit of the eigenvector* $\psi_{n,1,\beta} \to \psi_{n,1,0}$ *as* $b \to 0^+$, *for* $\beta \in \Omega_n$, *at* θ *fixed. Thus, we have the limit* $\psi_{n,1,\beta}(z) \to \psi_{n,1,0}(z)$ *as* $b \to 0^+$, *for* $\beta \in \Omega_n$, θ *fixed, uniformly for* z *on a compact set of the complex plane.*

Proof. Since the perturbed eigenfunctions are entire, as the unperturbed ones, we have the stability of the n zeros of $\psi_{n,1,0}(z)$ for b small.

For any fixed regular closed curve $\gamma = \partial\Gamma$ on the complex plane, oriented in the positive sense, turning around the short Stokes line, $S = [x_-, x_+]$, where $x_\pm = \pm\sqrt{E_n(1,0)} = \pm\sqrt{(2n+1)}$, we have the constant number of zeros in Ω_n (as in the phase-integral quantization):

$$n = \frac{1}{2\mathrm{i}\pi}\oint_\gamma \frac{\psi'_{n,1,\beta}(z)}{\psi_{n,1,\beta}(z)}\,\mathrm{d}z = \frac{1}{2\mathrm{i}\pi}\oint_\gamma \frac{\psi'_{n,1,0}(z)}{\psi_{n,1,0}(z)}\,\mathrm{d}z,$$

for $\beta \in \Omega'_n$, where

$$\Omega'_n = \{\beta \in \mathbb{C};\ 0 < b \leqslant b'_n,\ |\theta| \leqslant \pi\},$$

and $0 < b'_n \leqslant b_n$. Let us set $\psi_\beta = \psi_{n,1,\beta}$ and $\psi_0 = \psi_{n,1,0}$ and apply the theorem of Rouché [18]. Since the zeros of $\psi_0(z)$ are not on γ, there exists $M > 0$, such that $|\psi_0| \geqslant M > 0$ uniformly on γ. Moreover, $|\psi_\beta(z) - \psi_0(z)| \to 0$ uniformly for z on the compact γ, because of the analyticity. Thus, we have $|\psi_0(z)| > |\psi_\beta(z) - \psi_0(z)|$ for z on γ and for $\beta \in \Omega'_n \bigcup\{0\}$, so that the Rouché theorem applies. □

We shall see (theorem 2) that, for $\beta \in \Omega'_n$, the n nodes are confined on $\Gamma \bigcap \mathbb{C}_-$, where $\mathbb{C}_- = \{z \in \mathbb{C};\ \Im(z) < 0\}$, moreover, they are the only zeros of $\psi(z)$ on $\mathbb{C}_-$.

3. The confinement of the nodes on a half plane

Let us consider the spectral equivalence for scaling $x \to x/\sqrt{b}$:

$$H_{1,1}(\beta) \sim b^{-1} H_{\hbar,1}(\beta/b) = \hbar^{-1} H_{\hbar,1}(\beta') = \hbar^{-1} H_\hbar(\theta),$$

where

$$H_{\hbar,\alpha}(\beta) = \hbar^2 p^2 + \alpha x^2 + \mathrm{i}\sqrt{\beta}x^3 \tag{8}$$

is the semiclassical three-parameter Hamiltonian, and

$$\beta' = \hbar^{-1}\beta = b^{-1}\beta = \exp(\mathrm{i}\theta).$$

We have the identity of the eigenvalues:

$$E = E_{n,1}(\beta) = \hbar^{-1} E_{n,\hbar,1}(\hbar^{-1}\beta) = \hbar^{-1} E_n n, \hbar(\theta) = \hbar^{-1} E',$$

and the four-parameter wavefunction is

$$\psi_{n,\hbar,\theta}(x) = b^{-1/4}\psi_{n,1,1,\beta}(x/\sqrt{b}).$$

From now on, we redefine the parameters as

$$H_{\hbar,1}(\beta) = H_\hbar(\theta), \qquad E_{n,\hbar,1}(\beta) = E_{n,\hbar}(\theta), \tag{9}$$

$$\psi_{n,\hbar,1,\beta}(x) = \psi_{n,\hbar,\theta}(x), \tag{10}$$

where $\beta = \exp(\mathrm{i}\theta)$, $-\pi \leqslant \arg(\beta) = \theta \leqslant 0$, (the condition $0 \leqslant \arg(\beta) = \theta \leqslant \pi$ is equivalent). The stability result of perturbation theory for an $n \in \mathbb{N}$, $\hbar = b = |\beta|$ small, implies

$$|E = E_{n,\hbar}(\theta)| = O(\hbar), \qquad |z_{j,n}|, |z_{\pm}(E)| = O(\sqrt{\hbar}),$$

where $\{z_{j,n}\}_j$ are the stable n nodes of $\psi_{n,\hbar,\theta}(x)$. We consider the semiclassical operator

$$H_\hbar(\theta) = \hbar^2 p^2 + x^2 + \mathrm{i}\exp(\mathrm{i}\theta/2)x^3 = \hbar^2 p^2 + x^2 + \mathrm{i}\sqrt{\beta'}x^3,$$

and the eigenvalue $E_{n,\hbar}(\theta)$, with eigenfunction $\psi_{n,\hbar,\theta}$, where $n = 0, 1, \ldots$ and $|\beta| = 1$. We call $z = x + \mathrm{i}y$ the x variable extended to the complex plane. We consider the eigenvalues $E = E_{n,\hbar}(\theta)$, and the eigenfunctions $\psi_E(z) = \psi_{n,\hbar,\theta}(z)$, where the label n is the number of zeros stable at $\hbar = 0$.

We now prove the existence of a strip free from zeros of the eigenfunctions (see also [6]):

Theorem 2. *On the strip,*

$$Z(\theta) = \left\{ z \in \mathbb{C}; 0 \leqslant \Im(z) \leqslant y_+ = y_+(\theta) = \frac{2\Re\sqrt{\beta'}}{3} = \frac{2}{3}\cos\left(\frac{\theta}{2}\right)\right\},$$

there are no zeros of any eigenfunction $\psi(z)$ *of* $H_\hbar(\theta)$, *where* $\theta = \arg(\beta)$, $-\pi \leqslant \theta \leqslant 0$.

Proof. Let us, at first, set $-\pi < \theta \leqslant 0$, and consider the translated operator $H_{\hbar,y}(\theta) = \hbar^2 p^2 + V_y$, where

$$\begin{aligned} V_y = V_y(x) &= (x+\mathrm{i}y)^2 + \mathrm{i}\sqrt{\beta'}(x+\mathrm{i}y)^3 \\ &= x^2 - y^2 - 3\sqrt{\beta'}yx^2 + \sqrt{\beta'}y^3 + 2\mathrm{i}yx - 3\mathrm{i}\sqrt{\beta'}y^2x + \mathrm{i}\sqrt{\beta'}x^3. \end{aligned}$$

Let $\psi_y(x) = \psi_E(x+\mathrm{i}y)$ be an eigenfunction with eigenvalue E. We have

$$\psi_E(x+\mathrm{i}y) = \psi_y(x) \neq 0, \qquad \|\psi_y\| = 1,$$

for every $x \in \mathbb{R}$, for $0 \leqslant y \leqslant y_+ = 2\Re(\sqrt{\beta'})/3$.

For $0 \leqslant y \leqslant y_+$,

$$-\hbar^2\Im\left(\psi_y(r)\frac{\mathrm{d}\overline{\psi_y(r)}}{\mathrm{d}r}\right) = \int_r^\infty \Im(V_y(x) - E)|\psi_y(x)|^2\,\mathrm{d}x > 0, \tag{11}$$

or

$$-\hbar^2\Im\left(\psi_y(r)\frac{\mathrm{d}\overline{\psi_y(r)}}{\mathrm{d}r}\right) = -\int_{-\infty}^r \Im(V_y(x) - E)|\psi_y(x)|^2\,\mathrm{d}x > 0, \tag{12}$$

for any $r \in \mathbb{R}$. □

The proof is based on the monotonicity of

$$f(x) = \Im(V_y(x) - E) = R(x^3 - 3y^2x) + 2xy - 3Iyx^2 + c,$$

where $R = \Re(\sqrt{\beta})$, $I = \Im(\sqrt{\beta'})$ and c is a constant, that is, the non-negativity of $f'(x)$,

$$f'(x) = \Im(V_y(x) - E)' = 3Rx^2 - 6Iyx - 3Ry^2 + 2y = Ax^2 + Bx + C \geqslant 0,$$

where $A = 3R, B = -6Iy, C = -3Ry^2 + 2y$. We impose the non-positivity of the discriminant:

$$B^2 - 4AC = 12y[3y - 2R] \leqslant 0,$$

proved for

$$0 \leqslant y \leqslant y_+ = \frac{2R}{3} = \frac{2}{3}\cos\left(\frac{\theta}{2}\right).$$

We have the absence of zeros for $0 \leqslant \Im z \leqslant y_+$.

In the case of $\arg(\beta) = -\pi$, we have the limits of equations (11), (12), and for any $r \in \mathbb{R}$,

$$-\hbar^2 \Im\left(\psi(r)\frac{\mathrm{d}\overline{\psi(r)}}{\mathrm{d}r}\right) = \int_r^\infty \Im(V(x) - E)|\psi_E(x)|^2\,\mathrm{d}x = -\int_r^\infty \Im(E)|\psi_E(x)|^2\,\mathrm{d}x > 0,$$

$$\text{or} \quad \int_{-\infty}^r \Im(E)|\psi_E(x)|^2\,\mathrm{d}x > 0, \tag{13}$$

if $\Im(E) \neq 0$. Thus, the nodes have imaginary part different from zero, if the imaginary part of the eigenvalue is different from zero. Following is the case.

Lemma 2. *An eigenvalue $E(\theta) = E_{n,\hbar}(\theta)$, $n \in \mathbb{N}$, $\theta = -\pi$, $b > 0$, of $H_\hbar(\beta)$, has a negative imaginary part: $\Im E(-\pi) < 0$. On the other side, for $\theta = \pi$, we have $\Im E(\pi) > 0$.*

Proof. For $\arg(\beta) = -\pi$ fixed and $-r$ large, the normalized wavefunction ψ_E, $\|\psi_E\| = 1$ satisfies the Gamow semiclassical condition. This means that the eigenfunction is proportional to the Gamow solution

$$\psi_E = c_-\psi_-, \qquad \text{where} \quad c_- \neq 0,$$

since $\psi_E \neq 0$. The Gamow solution is defined by

$$\hbar\psi_-'(r)/\psi_-(r) \to -\mathrm{i}\sqrt{-V(r)}, \sqrt{-V(r)}|\psi_-(r)|^2 \to 1,$$

as $r \to -\infty$, and is the continuation of the subdominant solution in the Stokes sector $S_1(-\pi)$ to its upper border line $\arg(z) = -\pi$. Thus, we have,

$$-\Im(E) = -\frac{\hbar^2}{\int_r^{+\infty}|\psi_E(x)|^2\,\mathrm{d}x}\Im\left(|\psi_E(r)|^2\frac{\psi_E'(r)}{\psi_E(r)}\right)$$

$$\sim \frac{\hbar}{\int_r^{+\infty}|\psi_E(x)|^2\,\mathrm{d}x}\sqrt{-V(r)}|\psi_E(r)|^2 \to \hbar|c_-|^2 > 0, \tag{14}$$

as $r \to -\infty$, implying $\Im(E) < 0$. □

In particular, we know that the width of these 'resonances' is exponentially small: $-\Im(E_n) = O(\exp(-A(0))/\hbar)$, for any fixed $n \in \mathbb{N}$, where

$$A(0) = 2\int_{x_0}^{x_-}\pi_0(x)\,\mathrm{d}x = 2\int_{-1}^0\sqrt{x^2 + x^3}\,\mathrm{d}x = \frac{8}{15} \tag{15}$$

is the action of the forbidden classical motion on the barrier at $E = 0$.

In the general case, the nodes stay on the half plane:

$$\mathbb{C}_- = \{z \in \mathbb{C}; \Im(z) < 0\}.$$

Theorem 3. *For small $\hbar > 0$, $-\pi \leqslant \theta \leqslant 0$, all the stable n nodes of an eigenfunction $\psi_E(z)$ of $H_\hbar(\theta)$, with eigenvalue $E = E_{n,\hbar}(\theta)$, are in C_-.*

Proof. Let now $-\pi < \theta \leqslant 0$. We have the result from theorem 1 and theorem 2. The case $\theta = -\pi$ is proved by (13) and lemma 2. □

Definition 1. *We define the half plane*

$$\mathbb{C}^+_-(\theta) = \left\{z \in \mathbb{C};\, \Im(z) \leqslant y_+(\theta) = \tfrac{2}{3}\cos(\theta/2)\right\},$$

the half plane containing only the stable zeros $\{z_{n,j}\}_j$ *of the eigenfunction* $\psi_{n,\hbar,\theta}(z)$*, for any* $n \in \mathbb{N}$, $\hbar > 0$, $|\theta| \leqslant \pi$.

We now give a confinement of the nodes on the lower half plane.

Let $f = (1/\sqrt{\beta'}) = \exp(-\mathrm{i}\theta/2)$, $g = fE$, for $y < 0$,

$$-\hbar^2\Im\left(-\mathrm{i}f\psi_y(r)\frac{\mathrm{d}\overline{\psi_y(r)}}{\mathrm{d}r}\right) = \int_r^\infty \Im(-\mathrm{i}fV_y(x) + \mathrm{i}g)|\psi_y(x)|^2\,\mathrm{d}x < 0 \quad (16)$$

for $x > x_+$ and

$$-\hbar^2\Im\left(-\mathrm{i}f\psi_y(r)\frac{\mathrm{d}\overline{\psi_y(r)}}{\mathrm{d}r}\right) = -\int_{-\infty}^r \Im(-\mathrm{i}fV_y(x) + \mathrm{i}g)|\psi_y(x)|^2\,\mathrm{d}x > 0 \quad (17)$$

for $r < x_-$. Setting $f = R + \mathrm{i}I$, $R = \cos(\theta/2) > 0$, $I = -\sin(\theta/2) \geqslant 0$, for $-\pi < \theta \leqslant 0$, $\Re(g) = P > 0$ we have

$$\Im(-\mathrm{i}fV_y(x) + \mathrm{i}g) = (3y - R)x^2 + 2Iyx + y^2(R - y) + P < 0$$

for $x > x_+(y)$, $x < x_-(y)$, where

$$x_\pm(y) = \frac{1}{R - 3y}\left(Iy \pm \sqrt{I^2y^2 + (y^2(R - y) + P)(R - 3y)}\right).$$

We have $x_\pm(0^-) = \sqrt{P/R}$ $(x_\pm(y)/y) \to \pm(1/\sqrt{3})$ for $-y \to \infty$. For n, θ fixed, $E = E_{n,\hbar}(\theta) = O(\hbar)$, $P = O(\hbar)$,

$$x_\pm(0^-) = O(\sqrt{\hbar}). \quad (18)$$

4. The semiclassical limitation of the nodes for small parameter

Let us consider the semiclassical three-parameter Hamiltonian (8)

$$H_{\hbar,\alpha}(\beta') = \hbar^2p^2 + \alpha x^2 + \mathrm{i}\sqrt{\beta'}x^3, \quad (19)$$

with eigenvalues and eigenfunction

$$E_{n,\hbar,\alpha}(\beta'),\ \psi_{n,\hbar,\alpha,\beta'}(x) \quad (20)$$

where $\beta' = \exp(\mathrm{i}\theta)$, $\beta = \hbar\beta'$, $-\pi \leqslant \arg(\beta) = \theta \leqslant 0$ (the condition $0 \leqslant \arg(\beta) = \theta \leqslant \pi$ is equivalent).

Fixing a value of E, we get a Stokes complex [6]. A part of this complex is stable at $\beta = 0$. In particular, for the unperturbed parameters,

$$\alpha = 1, \quad \beta = 0, \quad E > 0,$$

we define the real line as the line of classical motion l_c.

We divide the line of classical motion in the short Stokes line of allowed motion $S = S(E) = [x_-, x_+]$, $x_\pm = x_\pm(E) = \pm\sqrt{E}$ and in the two anti-Stokes lines $A_\pm = A_\pm(E) = (x_\pm, \pm\infty)$.

It is easy to see that the line l_c is continuous in E for $E \in \mathbb{C}_c$, locally stable in β at $\beta = 0$, and locally continuous for $\beta \in \mathbb{C}_c$.

Now, we make the hypothesis that all the eigenvalues are perturbative. This hypothesis will be proved later.

We fix $n \in \mathbb{N}$.

We consider $E = E_{n,\hbar}(\theta)$, $\psi_{n,\hbar,\theta}(z)$ for $z \in \mathbb{C}_-$, $\beta = \hbar \exp(\mathrm{i}\theta) \in \Omega'_n$. In this case we know that there are n stable zeros (nodes) in $\mathbb{C}_-$. The question is whether there are other zeros.

We want to prove the absence of zeros in $\mathbb{C}_-$ for large $|z|$, uniformly for $\beta \in \Omega'_n$. Moreover, we prove that, in this semiclassical scaling, the zeros vanish as $\hbar \to 0$. This means that the zeros of $\psi_{n,\hbar,\theta}(z)$ on $\mathbb{C}_-$, for $\beta \in \Omega'_n$, are the nodes.

The eigenfunction $\psi(z) = \psi_{n,\hbar,\theta}(z)$ is an entire function and

$$(\psi(z), \psi'(z)) \to 0$$

as $|z| \to \infty$, for $\arg(z)$ in each of the two Stokes angular sectors $S_{\pm 1}(\theta)$.

On the other side, $\psi(z)$ is purely divergent in the other three sectors $S_0(\theta)$, $S_{\pm 2}(\theta)$, and has no zeros [14] in the full angular sector of the complex plane

$$S = S(\theta) = S_{-2} \bigcup \bar{S}_{-1} \bigcup S_0 \bigcup \bar{S}_1 \bigcup S_2 = \left\{ z \in \mathbb{C};\left|\arg(\mathrm{i}z) - \frac{\theta}{10}\right| < \pi \right\}$$

for large $|z|$. This fact is uniform for $\beta = \hbar \exp(\mathrm{i}\theta)$ in the compact Ω'_n.

Let us note that $S_- = \overline{\mathbb{C}}_- \subset S(\theta)$ for all θ, $-\pi \leqslant \theta \leqslant 0$.

We know that $|E_{n,\hbar}(\theta)| = O(\hbar)$ as $\hbar \to 0$, for $n \in \mathbb{N}$ fixed. Let us consider the anti-momentum

$$\pi_E(z) = \pi_{\hbar,\theta}(z) = \sqrt{V(\theta, z) - E_{n,\hbar,\theta}}, \tag{21}$$

where $V(\beta, z) = z^2 + \mathrm{i}\sqrt{\beta} z^3$, $\beta = \hbar \exp(\mathrm{i}\theta)$. There are three zeros of $\pi_E(z)$, the stable turning points $z_\pm(E)$ $|z_\pm(E)| = O(\sqrt{\hbar})$, for $E = O(\hbar)$, and $z_0(E)$.

For $-\pi < \theta \leqslant 0$, $\Im(z_0(E)) > 0$ for $E = O(\hbar)$, $\hbar$ being small enough.

The choice of the sign of $\pi_E(z) \neq 0$ is unique for $z \in \mathbb{C}_-$, and is defined by the action integral $\int_{z_+}^{z} \pi_E(z)\,\mathrm{d}z > 0$, along the anti-Stokes line A_+. We have

$$f = f_{\hbar,\theta}(z) \to 1, \qquad \text{where} \quad f(z) = \frac{\hbar|\psi'(z)|}{|\pi_E(z)||\psi(z)|}, \tag{22}$$

for any θ, $-\pi < \theta \leqslant 0$, $z \in \mathbb{C}_-$, $|z| > 0$, or for $\theta = -\pi$ $z \neq -1$, $|z| > 0$ and $E = O(\hbar)$ as $\hbar \to 0$.

Since there are no double zeros of the solution of the Schrödinger equation $\psi(z)$, none of the zeros of $\psi(z)$ goes to (or comes from) infinity on the sector $|\arg(\mathrm{i}z)| \leqslant \pi/2$, for $\beta \in \Omega'_n$. Furthermore, for $-\pi < \theta \leqslant 0$, all the zeros in $\mathbb{C}_-$ vanish as $\hbar \to 0$.

Remark 1. In the case $\theta = -\pi$, there is a problem: $z_0 \to -1$, $\pi_E(-1) \to 0$, as $\hbar \to 0$, $E \to 0$.

But this does not mean the presence of zeros on $\mathbb{C}_-$, near $z = -1$ for $\hbar$ small. This point is not trapping for the states of the problem.

We know that, for $n \in \mathbb{N}$ and $0 \geqslant \theta > -\pi$ fixed, and small $\hbar > 0$, the only zeros in $\mathbb{C}_-$ are near the origin.

We prove that these zeros are not able to cross a barrier in order to approach $z = -1$ as $\theta \to -\pi$.

The barrier is on the half circle $\gamma = \{z \in \mathbb{C}; z + 1 = \epsilon \exp(\mathrm{i}\phi), -(\pi) \leqslant \phi \leqslant (0)\}$ for any fixed ϵ, $1 > \epsilon > 0$. For $n \in \mathbb{N}$ fixed, we have $f_{\hbar,\theta}(z) \to 1$, uniformly for $z \in \gamma$, $\theta \in [-\pi, 0]$, as $\hbar \to 0^+$. This means that for $\hbar > 0$ small, no zeros from a neighborhood of the origin can reach a neighborhood of $z = -1$ as $\theta \to -\pi^+$.

Thus, we have

Theorem 4. *Let $n \in \mathbb{N}$ be fixed. $\beta = \hbar \exp(\mathrm{i}\theta) \in \Omega'_n$, $E = E_{n,\hbar}(\theta)$, $-\pi \leqslant \theta \leqslant 0$. The n zeros $\{z_{n,j}\}_j$ of the eigenfunction $\psi(z) = \psi_{n,\hbar,\theta}(z)$ in the half plane S_- are stable for $\beta = \hbar \exp(\mathrm{i}\theta) \in \Omega'_n$, and vanish as $O(\sqrt{\hbar})$ for $\hbar \to 0$.*

5. The semiclassical limitation of the nodes for large parameter

Let us consider the three-parameter operators

$$H_{\hbar,\alpha}(\beta) = -\hbar^2 p^2 + \alpha x^2 + \mathrm{i}\beta x^3,$$

and the eigenvalues $E_{n,\hbar,\alpha}(\beta)$, for $n \in \mathbb{N}$. With a suitable scaling $\hbar = \beta = 1$, and,

$$H_\alpha = H_{1,\alpha}(1) = p^2 + V_\alpha = p^2 + \alpha x^2 + \mathrm{i}x^3,$$

with an eigenvalue and eigenfunction:

$$E = E_\alpha = E_{n,1,\alpha}(1), \qquad \psi_\alpha(z) = \psi_{n,1,\alpha,1}(z)$$

for fixed $n = 0, 1, \ldots$ and

$$\alpha \in B_n, \qquad \text{where} \qquad B_n = \{\alpha \in \mathbb{C};\, \alpha = 0, \text{ or } 0 < a = |\alpha| \leqslant a_n, |\arg(\alpha)| \leqslant 2\pi/5\},$$

where $a_n = (1/b'_n)^{2/5} > 0$ and b'_n is given in lemma 1. We have, $E \neq 0$, $|\arg(E)| \leqslant \pi/2$, because of the numerical range and the uncertainty principle.

The eigenfunction $\psi_\alpha(z)$ is an entire function and,

$$(\psi_\alpha(z), \psi'_\alpha(z)) \to 0$$

as $|z| \to \infty$, for $\arg(z)$ in each of the two Stokes angular sectors $S_{\pm 1}(0)$.

On the other side, $\psi_\alpha(z)$ is purely divergent in the other three sectors S_0, $S_{\pm 2}$, and has no zeros [14] in the full angular sector of the complex plane

$$S = S(0) = S_{-2} \bigcup \bar{S}_{-1} \bigcup S_0 \bigcup \bar{S}_1 \bigcup S_2 = \{z \in \mathbb{C};\, |\arg(\mathrm{i}z) < \pi\}$$

for large $|z|$.

We have the following result.

Theorem 4′. *Let $E_\alpha = E_{n,1,\alpha}(1)$ be the simple eigenvalue for fixed $n = 0, 1, \ldots, \alpha \in B_n$. Then, none of the nodes of its eigenfunction $\psi_\alpha(z) = \psi_{n,1,\alpha,1}(z)$ goes to (or comes from) infinity on the sector*

$$S_- = \{z \in \mathbb{C};\, |\arg(\mathrm{i}z)| \leqslant \pi/2\} = \overline{\mathbb{C}}_-.$$

Remark 2. Considering also theorem 2, we have the invariance of the number of nodes.

Proof of theorem 4′. Let us consider the function

$$f_\alpha(z) = \frac{|\psi'_\alpha(z)|}{|\pi_\alpha(z)||\psi_\alpha(z)|}, \tag{23}$$

where $|\pi_\alpha(z)| = \sqrt{|V_\alpha(z) - E_\alpha|}$ diverge as $|z| \to \infty$, $|\arg(\mathrm{i}z)| \leqslant \pi/2$, for any fixed $\alpha \in B_n$, $E_\alpha \in \mathbb{C}$.

Since $\psi_\alpha(z)$ is the analytic solution of the Schrödinger equation, with energy E_α, the zeros of $\psi_\alpha(z)$ are simple and $f_\alpha(z)$ diverge where $\psi_\alpha(z)$ has a zero. We have the semiclassical behavior of $\psi_\alpha(z)$ for large $|\pi_\alpha(z)|$, so that

$$f_\alpha(z) \to 1$$

as $|z| \to \infty$ uniformly for $|\arg(\mathrm{i}z)| \leqslant \pi/2$, $\alpha \in B_n$, $E_\alpha \in \mathbb{C}$. This means that no zero of $\psi_\alpha(z)$ goes to (or comes from) infinity on the sector $|\arg(\mathrm{i}z)| \leqslant \pi/2$, for this set of parameters. □

6. Limitation of the eigenvalues

We use the same scaling as the previous section and our Hamiltonian

$$H_\alpha = H_{1,\alpha}(1) = p^2 + V_\alpha = H_\alpha(1) = p^2 + \alpha x^2 + \mathrm{i}x^3,$$
$$E = E_\alpha = E_{n,1,\alpha}(1)$$

for fixed $n = 0, 1, \ldots$. is an eigenvalue with eigenfunction

$$\psi_\alpha(z) = \psi_{n,1,\alpha,1}(z)$$

for fixed $\alpha \in B_n$. Fixing a value of E, we get a Stokes complex [6] as above.

We prove the limitation of the eigenvalues $E_{n,\alpha}(1)$ for bounded parameters (n, α). In particular, $n = 0, 1, \ldots$ is fixed, $|\alpha| \in [0, a_n]$, where $a_n = (1/b'_n)^{2/5} > 0$, $|\arg(\alpha)| < 2\pi/5$.

This result forbids the disappearance or appearance of an eigenvalue $E_{n,1}(\beta)$ at infinity at a fixed $\beta \in \overline{\mathbb{C}}_c$. For our non-self-adjoint operators, we use an argument slightly different from the one of [1]. We directly use the semiclassical quantization and the stability of the nodes.

Theorem 5. *For any fixed $n = 0, 1, \ldots$ and $\alpha, \widehat{\alpha} \in B_n$, $E_\alpha = E_{n,1,\alpha}(1)$ is bounded and continuous at $\alpha = \hat{\alpha}$.*

Proof. Let us consider the three-parameter operators

$$H_{1,\alpha}(\beta') = \hbar^2 p^2 + \alpha x^2 + \mathrm{i}\sqrt{\beta'}x^3,$$

and the eigenvalues $E_{n,1,\alpha}(\beta')$ for $n \in \mathbb{N}$. We have the spectral equivalence for positive scaling:

$$H_{1,\alpha}(1) \sim H_{\lambda^{-1},\lambda^2\alpha}(\lambda^6),$$

so that

$$E_{n,1,\alpha}(\beta')(1) = E_{n,\lambda^{-1},\lambda^2\alpha}(\lambda^6),$$

for $n \in \mathbb{N}$, $\lambda > 0$.

Because of the analyticity of the family of operators $H_\alpha = H_{1,\alpha}(1)$, the limitation of the eigenvalue $E_\alpha = E_{n,1,\alpha}(1)$ implies its continuity.

We prove the limitation by absurd. □

Let us fix $n \in \mathbb{N}$, and $\hat{\alpha}$, $0 \leqslant \hat{a} = |\widehat{\alpha}| \leqslant a_n$, $|\arg(\hat{\alpha})| \leqslant 2\pi/5$ for $\hat{a} \neq 0$, and suppose $|E_\alpha = E_{n,1,\alpha}(1)| \to \infty$ as $\alpha \to \widehat{\alpha}$.

For α near $\widehat{\alpha}$, we scale the Hamiltonian and use the identity:

$$\lambda^{6/5} E_{n,1,\alpha}(1) = E_{n,\hbar,\alpha'}(1) = E_{\alpha'} := s = \exp(\mathrm{i}\phi),$$

where

$$s = s(\alpha), \qquad \lambda = \hbar = |E_{n,1,\alpha}(1)|^{-5/6} > 0, \qquad \alpha' = \lambda^{2/5}\alpha, \qquad |\phi| < \pi/2.$$

We set

$$s_0 = s(\hat{\alpha}), \qquad |s_0| = 1.$$

Thus, we study the semiclassical eigenvalue problem $H_{\alpha'}\psi_{\alpha'} = s\psi_{\alpha'}$, where $H_{\alpha'} = H_{\hbar,\alpha'}(1)$, by the Bohr–Sommerfeld quantization rule (semiclassical phase-integral quantization). For small $\hbar$, we have

$$n = \frac{1}{2\mathrm{i}\pi}\oint_\gamma \frac{\psi'_{\alpha'}(z)}{\psi_{\alpha'}(z)}\,\mathrm{d}z = \mathrm{i}\frac{1}{2\pi\hbar}\oint_\gamma \pi_s(z)\,\mathrm{d}z - \frac{1}{2} + O(\hbar), \tag{24}$$

where n is the number of nodes.

Remark 3. This quantization rule is obtained by the WKB approximation on the line of classical motion for $z \in A_+ = A_+(E)$, with the choice of the subdominant solution,

$$\frac{\psi'_{\alpha'}(z)}{\psi_{\alpha'}(z)} = -\frac{1}{\hbar}\pi_s(z) - \frac{1}{2}\frac{\pi_s(z)'}{\pi_s(z)} + O(\hbar),$$

continued at the hedges of the cut on the short Stokes line $S(s)$.

Thus, the Bohr–Sommerfeld quantization rule reads

$$J_\alpha(s) = \mathrm{i}\oint_\gamma \pi_s(z)\,\mathrm{d}z = 2\int_{S(s)} p_s(z)\,\mathrm{d}z = \pi(2n+1)\hbar + O(\hbar^2), \tag{25}$$

where the phase of $\pi_s(z) = \sqrt{V(z) - s}$ is defined on the anti-Stokes line A_+, so that $\int_{z_+}^{z} \pi_s(z)\,\mathrm{d}z > 0$ along A_+, and $\int_{z_-}^{z} p_s(z)\,\mathrm{d}z > 0$ along $S(s)$ by definition.

The two turning points have the limit values $z_+ \to (-\mathrm{i}s)^{1/3}$, $z_- \to z_+\exp(-2\mathrm{i}\pi/3)$ as $\alpha' \to 0$, so they are confined on the fixed compact domain $\Gamma \in \mathbb{C}_-$ with the n nodes (see theorem 2). The path $\gamma = \partial\Gamma$ is oriented in the positive sense.

The limit $\alpha \to \hat{\alpha}$ implies $s \to s_0$, $\alpha', \hbar \to 0$, and

$$\begin{aligned} J_\alpha(s) \to J_{\widehat{\alpha}}(s_0) &= \mathrm{i}\oint_{\gamma'} \sqrt{\mathrm{i}z^3 - \widehat{\epsilon}}\,\mathrm{d}z = s_0^{5/6}\mathrm{i}\oint_{\gamma''}\sqrt{\mathrm{i}y^3 - 1}\,\mathrm{d}y \\ &= s_0^{5/6}\oint_{\gamma''}\sqrt{1 - \mathrm{i}y^3}\,\mathrm{d}y = s_0^{5/6}2\Re\left(2\exp(-\mathrm{i}\pi/6)\int_0^1\sqrt{1-x^3}\,\mathrm{d}x\right) \\ &= s_0^{5/6}4\sin\left(\frac{\pi}{3}\right)\int_0^1\sqrt{1-x^3}\,\mathrm{d}x \\ &= s_0^{5/6}2\sqrt{\pi}\sin\left(\frac{\pi}{3}\right)\frac{\Gamma(1+(1/3))}{\Gamma((1/3)+(3/2))} \neq 0, \end{aligned} \tag{26}$$

where $y = z(s_0)^{-2/3}$, and where the phase of $\sqrt{iz^3 - s_0}$ vanishes as $|z| \to \infty$, for $\arg(z) = -\pi/6$, and where γ, in this semiclassical approximation, has been distorted to a regular path γ' encircling the origin and both the turning points $z_\pm$, and rescaled to the path γ''.

As a result, for the left-hand side of equation (25) we have

$$J_\alpha(s) \to 0, \tag{27}$$

as $\alpha \to \widehat{\alpha}$, $\hbar \to 0$, $s \to s_0$, in contradiction with the limit of the left-hand side of equation (25), as written in equation (26). The proof is similar for $\widehat{\alpha} = 0$. Let us note that the same analysis gives the correct semiclassical behavior of the eigenvalues [7], [5], for large n. From equations (26) and (25), we have

$$s_0^{5/6}2\sqrt{\pi}\sin\left(\frac{\pi}{3}\right)\frac{\Gamma(1+(1/3))}{\Gamma((1/3)+(3/2))} \sim \pi(2n+1)\hbar,$$

where

$$s_0 = E_n(\hbar, 0, 1) \to \left(\frac{\Gamma[(3/2)+(1/3)]\hat{J}}{2\sqrt{\pi}\sin(\pi/3)\Gamma[1+(1/3)]}\right)^{6/5},$$

as $n \to \infty$, $2n\hbar\pi \to \hat{J} > 0$ [7].

7. Absence of non-perturbative eigenvalues

We prove here that all the eigenvalues of our problem are perturbative. Our proof is based on the exclusion of all the possible sources of eigenvalues at $\hbar = 0$. Let us consider our semiclassical potential

$$V_f(z) = \frac{z^2(f + \mathrm{i}z)}{f}, \qquad \text{where} \quad f = \exp(-\mathrm{i}\theta/2) \neq 0.$$

Possible sources of eigenvalues are the stationary points acting as trapping points. The stationary points are defined by

$$V_f'(z) = \frac{z(2f + 3\mathrm{i}z)}{f} = 0,$$

giving the solutions

$$z = 0, \qquad z^d = \frac{2\mathrm{i}f}{3}.$$

We have

$$V_f(z^d) = \frac{(z^d)^2(f + \mathrm{i}z^d)}{f} = \frac{-4f^2}{27} := E^d(\theta), \qquad V_f''(z^d) = \frac{(2f + 6\mathrm{i}z^d)}{f} = -2 < 0.$$

The negativity of the second derivative allows us to call top resonances the possible levels with states concentrated near z^d as $\hbar \to 0$ with zeros along one of the Stoke lines defined by the condition:

$$\Im(\sqrt{\pm\mathrm{i}}(z - z^d)) = 0.$$

We shall prove that such levels are non-modal, that is, the states are not on the domain of the Hamiltonian. Actually, since z^d is on the upper boundary of the strip free of zeros of the modal solutions, a possible proof is the position of someone of such zeros for $\hbar$ small, $n = O(1/\hbar)$.

For completeness, we consider also the anti-bound states, concentrated near the origin, with zeros along the imaginary axis as $\hbar \to 0$. Such states are clearly non-modal.

Let us recall [14] the 5 Stokes angular sectors of the complex z plane,

$$S_k(\theta) = \left\{ z \in \mathbb{C}, z \neq 0; \left| \arg(\mathrm{i}z) + \frac{\theta}{10} - \frac{2k\pi}{5} \right| < \frac{\pi}{5} \right\},$$

$-2 \leqslant k \leqslant 2$.

Lemma 3. *The top resonances are non-modal.*

Proof. For $\hbar > 0$, $\beta' = \exp(\mathrm{i}\theta)$, $-\pi < \theta < \pi$, setting

$$z = z^d - \sqrt{\pm\mathrm{i}}\, y,$$

we get the Hamiltonians of top resonances

$$K_\hbar^\pm(\theta) = E^d(\theta) \mp \mathrm{i}(\hbar^2 p^2 + y^2 + \mathrm{i}\sqrt{\pm\beta'} y^3) = E^d(\theta) \mp \mathrm{i}H_\hbar(\theta \pm \pi/2)$$

defined on the domain of functions asymptotically vanishing in the two sectors

$$(S_{\mp 1}(\theta), \quad S_{\pm 2}(\theta)),$$

respectively. We remember [3] that we have stability of the eigenvalues for $|\theta| < 3\pi/2$, or $|\theta \pm \pi/2| < \pi$. The small $\hbar$ behavior of the eigenvalues, $E_{n,\hbar}(\theta \pm \pi/2) = \hbar(2n+1) + O(\hbar^2)$ of $H_\hbar(\theta \pm \pi/2)$, gives the behavior of the eigenvalues of $K_\hbar^\pm(\theta)$,

$$\mathcal{E}_{n,\hbar}^\pm(\theta) = E^d(\theta) \pm \hbar(2n+1) + O(\hbar^2).$$

The eigenfunctions, $\phi^{\pm}_{n,\hbar,\theta}(z)$ have nodes on the lines

$$\Im(\sqrt{\pm \mathrm{i}}(z - z^d)) = 0,$$

and for small $\hbar$ and $n = O(1/\hbar)$, about one half of the nodes are on the forbidden strip $0 \leqslant \Im(z) \leqslant y_+$. This means that the eigenfunctions are non-modal. □

Lemma 4. *The anti-bound states are non-modal.*

Proof. For $\hbar > 0$, $\beta' = \exp(\mathrm{i}\theta)$, $-\pi < \pm\theta \leqslant \pi/2$, setting,

$$z(y) = \pm \mathrm{i}y,$$

we get the Hamiltonians of anti-bound states,

$$\hat{H}^{\pm}_{\hbar}(\theta) = -(\hbar^2 p^2 + y^2 \mp \sqrt{\beta'} y^3) = -H_\hbar(\theta \pm \pi),$$

defined on the domain of functions vanishing at ∞ in the two sectors

$$(S_{\pm 2}(\theta), S_0(\theta)),$$

respectively. We remember [3] that we have stability of the eigenvalues for $|\theta| < 3\pi/2$, or $-\pi < \pm\theta < \pi/2$. The small $\hbar$ behavior of the eigenvalues $E_{n,\hbar}(\theta \pm \pi/2) = \hbar(2n+1) + O(\hbar^2)$ of $H_\hbar(\theta \pm \pi/2)$ gives the behavior of the eigenvalues of $\hat{H}^{\pm}_{\hbar}(\theta)$,

$$\hat{E}^{\pm}_{n,\hbar}(\theta) = -\hbar(2n+1) + O(\hbar^2).$$

The eigenfunctions $\phi^{\pm}_{n,\hbar,\theta}(z)$ have nodes on the line $\Re(z) = 0$, and for small $\hbar$ and $n = O(1/\hbar)$, about one half of the nodes are on the forbidden strip $0 \leqslant \Im(z) \leqslant y_+$. This means that the eigenfunctions are non-modal. □

Thus, Lemmata 3 and 4 prove the following theorem:

Theorem 1′. *For any fixed $n \in \mathbb{N}$, the eigenvalue $E_{n,1}(\beta)$, $\beta \in \mathbb{C}_c$, is simple and is given by the unique continuation of the eigenvalue of theorem 1. Its label n is the number of nodes of its eigenfunction, as it appears in the Bohr–Sommerfeld quantization rule.*

8. The power law behavior at infinity

We prove here the algebraic behavior of the eigenvalues for large parameter. We use the scaling formula

$$\sqrt{\alpha} E_{n,1}(\beta) = E_{n,\alpha}(1)$$

for $n \in \mathbb{N}$, where $\alpha = \beta^{-2/5}$. Let us recall that theorem 4, in the special case of $\alpha_0 = 0$, implies continuity and limitation of each eigenvalue $E_{n,\alpha}(1)$ in the limit $\alpha \to 0$.

The analyticity, of type A, of the family of operators $H_\alpha(1)$ (see [2] theorem 2.10), with the control of the nodes, and the simplicity of the spectrum imply the stability at $\alpha = 0$ and α−analyticity in a neighborhood of the origin of each eigenvalue $E_{n,\alpha}(1)$.

Therefore, if $\alpha = \beta^{-2/5}$, $\sqrt{\alpha} E_{n,1}(\beta) = E_{n,\alpha}(1) \to E_{n,0}(1)$ for $\beta \to \infty$. Thus, for b large, $E_{n,\alpha}(\beta)$ grows as $b^{1/5}$, and has an algebraic singularity there:

$$E_{n,1}(\beta) = \beta^{1/5} E_{n,\beta^{-2/5}}(1) \sim \beta^{1/5} E_{n,0}(1).$$

Let us note that we have $\arg(E_{n,1}(b\exp(\pm \mathrm{i}\pi))) \to \pm\pi/5$, and

$$\pm b^{-1/5} \Im(E_{n,1}(b \exp(\pm \mathrm{i}\pi))) \to E_{n,0}(1)\sin(\pi/5) > 0,$$

as $b \to \infty$.

9. Global analyticity, symmetry, and Padé summability on the cut plane

Let $E(\beta) = E_{n,1}(\beta), n = 0, 1, 2, \ldots$

$$f(\beta) = \frac{E(\beta) - E(0)}{\beta},$$

$f(\beta)$ is bounded holomorphic on the completed cut complex plane $\bar{\mathbb{C}}_c = \{\beta \in \mathbb{C}; \beta \neq 0, |\arg(\beta) = \theta| \leqslant \pi\}$ (see theorems 1, 2, 3′, 4). Moreover, we have the symmetry of the eigenvalues: $E_{n,1}(\beta) = \bar{E}_{n,1}(\bar{\beta})$, so that we have $f(\beta) = \bar{f}(\bar{\beta})$. By the Cauchy theorem, we have

$$f(\beta) = \frac{1}{2\mathrm{i}\pi}\oint_\gamma \frac{f(z)}{z-\beta} = \frac{1}{2\mathrm{i}\pi}\oint_\gamma \frac{1}{1-(\beta/z)}\frac{f(z)}{z}\,\mathrm{d}z = \int_0^\infty \frac{1}{(1+\beta x)}\rho(x)\,\mathrm{d}x,$$

where γ is any curve turning around β in the positive way. We have the dispersion relation of a Stieltjes function, where

$$\begin{aligned}\rho(1/b) &= -b(f(-b+\mathrm{i}0^+) - f(-b-\mathrm{i}0^+))/2\mathrm{i}\pi \\ &= -b\Im f(-b+\mathrm{i}0^+)/\pi = \Im E_n(-b+\mathrm{i}0^+)/\pi \geqslant 0,\end{aligned}$$

by lemma 1. We have the asymptotics to the formal power series:

$$f(\beta) \sim \Sigma(\beta) = \sum_{j=0}^{\infty} a_j(-\beta)^j$$

for $|\beta|$ small, where

$$a_j = |c_{j+1}| = \int_0^\infty x^j \rho(x)\,\mathrm{d}x \tag{28}$$

are the moments of the measure $\rho(x)\,\mathrm{d}x$. Thus, the moment problem

$$a_j = |c_{j+1}| = \int_0^\infty x^j \mathrm{d}\mu(x) \tag{29}$$

has the solution $\mathrm{d}\mu(x) = \rho(x)\,\mathrm{d}x$. Because of the bound on the perturbation coefficients $|c_j| < AC^j j!$ (see theorem 1 and [2, 3]), the Carleman theorem condition (see [22] page 330) is satisfied,

$$\sum_n (1/a_n)^{1/2n} = \infty,$$

and the uniqueness of the solution $\mathrm{d}\mu(x) = \rho(x)\,\mathrm{d}x$.

Let us recall the definition of the diagonal Padé approximants $R_n^n(\beta)$ of the formal power series $\Sigma(\beta) = \sum_{j=0}^{\infty} a_j(-\beta)^j$, with partial sums $\Sigma^N(\beta) = \sum_{j=0}^{N-1} a_j(-\beta)^j$, $\beta \in \mathbb{C}$. The diagonal Padé approximants, $R_n^n(\beta)$, $n \geqslant 0$, are the rational fractions

$$R_n^n(\beta) = \frac{P_n(\beta)}{Q_n(\beta)},$$

where $P_j(\beta)$, $Q_j(\beta)$ are the polynomials of degree j, with the condition $Q_j(0) = 1$, defined by the asymptotic condition, $\left|R_n^n(\beta) - \Sigma^{2n+1}(\beta)\right| = O(\beta^{2n+1})$, for $|\beta| \to 0$. As a general result, the Padé approximants $R_n^n(\beta)$ on Stieltjes asymptotic expansions do not have poles or zeros on the complex cut plane, and there converge

$$R_n^n(\beta) \to f_\mu(\beta) = \int_0^\infty \frac{1}{(1+\beta x)}\mathrm{d}\mu(x),$$

where $\mathrm{d}\mu$ is a measure solution of the moment problem (29). In this case, necessarily we have $\mathrm{d}\mu(x) = \rho(x)\,\mathrm{d}x$ and $f_\mu(\beta) = f(\beta)$.

Thus, we have the result:

Theorem 6. *The function*

$$f(\beta) = \frac{E(\beta) - E(0)}{\beta}$$

is a Stieltjes function,

$$f(\beta) = \int_0^\infty \frac{1}{(1+\beta x)} \rho(x)\,\mathrm{d}x, \tag{30}$$

for β on the cut complex plane, where

$$\rho(1/b) = \Im(E_n(-b + \mathrm{i}0^+))/\pi > 0, \tag{31}$$

and

$$\ln(\rho(x)) = -C^{-1}x(1 + O(\ln(x)/x)), \tag{32}$$

where $C^{-1} = 8/15 = 2B(2, 3/2) = A(0)$ (see (15) and [17]), for large positive x. The diagonal Padé approximants of the perturbation series, converge to f,

$$R_n^n(\beta) \to f(\beta),$$

as $n \to \infty$, uniformly for β on compacts of the cut complex plane.

Proof. The inequality (31) is proved by the $\mathcal{PT}$ symmetry of the eigenfunctions and eigenvalues $E_n(-b + \mathrm{i}0^+) - E_n(-b - \mathrm{i}0^+) = 2\mathrm{i}\Im(E_n(-b + \mathrm{i}0^+))$ and by lemma 2. We have only to discuss the asymptotic behavior of the discontinuity function.

For the semiclassical behavior of the discontinuity (32), we consider the semiclassical scaling where $b > 0$ plays the role of a semiclassical parameter, with the Gamow condition at $-\infty$:

$$H(b, -\pi) \sim bH(1, 1, b\exp(-\mathrm{i}\pi)).$$

In the case of the semiclassical operator $H(b, -\pi)$, we have a 'double-well problem', with the barrier action $C^{-1} = 8/15$ (15), and $\hbar = b$. This value of the barrier implies the behavior of $\rho(x)$, as $x \to \infty$, as given in (32), and the behavior of the perturbation coefficients $c_j = (-1)^j a_{j-1}$, $a_j = \int_0^\infty x^j \mathrm{d}\rho(x)$, $a_j \sim DC^j j!$, as $j \to \infty$, for some $D > 0$, compatible with the behavior:

$$c_j = (-1)^{j-1} 4\sqrt{15} C^j (2\pi)^{-3/2} \Gamma(j + 1/2)(1 + O(1/j)),$$

for large j, obtained numerically [9] in the case of $n = 0$. □

Remark 4. We have proved, in a new way, that each eigenvalue $E(\beta) = E_{n,1}(\beta)$, $n = 0, 1, 2, \ldots$, is simple and positive for positive β:

$$E(\beta) = E(0) + \beta f(\beta) = E(0) + \beta \int_0^\infty \frac{1}{(1+\beta x)} \rho(x)\,\mathrm{d}x \geqslant E(0),$$

and $E_{n,1}(\beta) \sim \beta^{1/5} E_{n,0}(1)$ for large positive β.

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